

LEFSCHETZ FIBRATIONS WITH SMALL SLOPE

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ABSTRACT. We construct Lefschetz fibrations over S^2 which do not satisfy the slope inequality. This gives a negative answer to a question of Hain.

1. INTRODUCTION

By the remarkable works of Donaldson [8] and Gompf [15], it turned out that Lefschetz fibrations are closely connected with symplectic 4-manifolds. As a result, the study of Lefschetz fibrations has been an active area of research. In this paper, we consider the geography problem of Lefschetz fibrations over S^2 which derives from that of complex surfaces fibred over curves.

We introduce two kinds of geography problems. Let σ and e be the signature and the Euler characteristic of a closed oriented smooth 4-manifold X , respectively. For an almost complex closed 4-manifold X , we set $K^2 := 3\sigma + 2e$ and $\chi_h := (\sigma + e)/4$ (the *holomorphic Euler characteristic*).

One is the geography problem for complex surfaces (i.e. the characterization of pairs (K^2, χ_h) corresponding to minimal complex surfaces. It is well-known that for a minimal complex surface of general type, $K^2 > 0$, $\chi_h > 0$ and $2\chi_h - 6 \leq K^2 \leq 9\chi_h$. The latter two inequalities are called the *Noether-* and *Bogomolov-Miyaoka-Yau-inequalities* (cf. [4]). The above geography problem can be extend to the symplectic 4-manifolds. However, there exists minimal symplectic manifolds which do not satisfy the Noether inequality. Fintushel and Stern [13] constructed Lefschetz fibration which does not satisfy the Noether inequality. In particular, for most pairs (p, q) satisfying $p < 2q - 6$, there exists a minimal symplectic 4-manifold with $p = K^2$ and $q = \chi_h$ (cf. [15]). On the other hand, no examples of a minimal symplectic 4-manifold with $K^2 > 9\chi_h$ have been found yet.

The other is the geography problem for complex surfaces fibred over curves. Hereafter, we assume $g \geq 2$. Let $f : S \rightarrow C$ be a relatively minimal holomorphic genus- g fibration, where S is a complex surface and C is a complex curve of genus k . We define relative numerical invariants $\chi_f := \chi_h - (g - 1)(k - 1)$ and $K_f^2 := K^2 - 8(g - 1)(k - 1)$ for $f : S \rightarrow C$. Then, we have two inequalities $\chi_f \geq 0$ and $K_f^2 \geq 0$ known as *Beauville's inequality* (cf. [3]) and *Arakelov's inequality* (cf. [2]), respectively. For $\chi_f \neq 0$, which is equivalent to the fact that f is not a holomorphic bundle, we define λ_f to be the quotient K_f^2/χ_f . We call λ_f the slope of f . Xiao [38] proved that $4 - 4/g \leq \lambda_f \leq 12$ (i.e., $(4 - 4/g)\chi_f \leq K_f^2 \leq 12\chi_f$). The former inequality is called the *slope inequality*.

The study of the slope of holomorphic fibrations was mainly motivated by Severi's inequality, which states that if S is a minimal surface of general type of maximal Albanese dimension, then $K^2 \geq 4\chi_h$. In another words, if $K^2 < 4\chi_h$, then S is a

surface fibred over C of genus $b_1(S)/2$. Severi [32] claimed it in 1932, but his proof was not correct (cf. [5]). The inequality was independently posed as a conjecture by Reid [30] and by Catanese [5]. Xiao [38] proved the conjecture when S is a surface fibred over a curve of positive genus. He showed that if S admits a holomorphic genus g fibration f over C of positive genus k with $K^2 < 4\chi_h + 4(g-1)(k-1)$ (i.e. $\lambda_f < 4$), then $k = b_1(S)/2$. Konno [21] proved the inequality in the case of *even* surfaces. The conjecture was solved by Manetti [24] when S has ample canonical bundle. Pardini [29] proved the conjecture completely by using the slope inequality for holomorphic fibrations over \mathbb{CP}^1 .

Let $f : X \rightarrow S^2$ be a relatively minimal genus- g Lefschetz fibration with n singular fibers. Then, χ_f , K_f^2 and the slope λ_f are defined in the same way as for complex surfaces fibred over curves. From $e(X) = -4(g-1) + n$ and the results of Ozbagci [28] and Stipsicz [35], we have $\chi_f \geq 0$, $K_f^2 \geq 4g - 4$ and $\lambda_f \leq 10$. By the result of Li [23], we find that $\chi_f = 0$ if and only if $n = 0$ (i.e., $X = \Sigma_g \times S^2$). Moreover, it is well-known that any hyperelliptic Lefschetz fibrations satisfy the slope inequality. Therefore, genus-2 Lefschetz fibrations satisfy the slope inequality. In particular, if f is a hyperelliptic Lefschetz fibration with only nonseparating vanishing cycles, then λ_f is equal to $4 - 4/g$. To author's knowledge, the slope of all known Lefschetz fibrations over S^2 is greater than or equal to $4 - 4/g$.

Conjecture 1.1 (Hain cf. [1], Question 5.10, [11], Conjecture 4.12). *For every genus- g Lefschetz fibration $f : X \rightarrow S^2$, the slope inequality $\lambda_f \geq 4 - 4/g$ holds.*

We can reformulate the slope inequality of Conjecture 1.1 in terms of the Deligne-Mumford compactified moduli space of stable curves of genus g , denoted by $\overline{\mathcal{M}}_g$, as follows. For a relatively minimal genus- g Lefschetz fibration $f : X \rightarrow S^2$ with n singular fibers, we can obtain a symplectic structure on X such that for all $x \in S^2$, $f^{-1}(x)$ is a pseudo-holomorphic curve. Since a 2-dimensional almost-complex structure is integrable, $f^{-1}(x)$ determines a point in $\overline{\mathcal{M}}_g$. Thus, we can obtain the moduli map $\phi_f : S^2 \rightarrow \overline{\mathcal{M}}_g$ which is defined by $\phi_f(x) = [f^{-1}(x)] \in \overline{\mathcal{M}}_g$ for $x \in S^2$. We denote by \mathcal{H}_g the Hodge bundle on $\overline{\mathcal{M}}_g$ with fiber the determinant line $\wedge^g H^0(C; K_C)$, where C is the set of critical points of f . By using Smith's signature formula [33], we have the following inequality which is equivalent to the slope inequality of Conjecture 1.1.

$$(8g + 4)\langle c_1(\mathcal{H}_g), [\phi_f(S^2)] \rangle - g \cdot n \geq 0$$

We give a negative answer to Conjecture 1.1.

Theorem 1.2. *For $g \geq 3$, there exist a genus- g Lefschetz fibration over S^2 with slope $\lambda_f = 4 - 1/g - 1/3g$ whose total space is simply connected and non-spin.*

Moreover, by fiber sum operations, we have the following results:

Corollary 1.3. *For each $g \geq 3$, $m \geq 0$ and $l \geq 0$, there a genus- g Lefschetz fibration $f_{m,l} : X_{m,l} \rightarrow S^2$ with slope $\lambda_{f_{m,l}} = 4 - 4/g - 1/(m+3)g$ s.t. $\pi_1(X_{m,l}) = 1$. Moreover, if $(m, l) \neq (0, 0)$ (resp. $l \not\equiv 0 \pmod{16}$), then $X_{m,l}$ is minimal (resp. non-spin) symplectic 4-manifold.*

Corollary 1.4. *For each $g \geq 3$, $m \geq 1$ and $l \geq 0$, there exist a genus- g Lefschetz fibration $f'_{m,l} : Y_{m,l} \rightarrow S^2$ with slope $\lambda_{f'_{m,l}} = 4 - 4/g - 1/2g + 1/(2 \cdot 3^{m-1}g)$ s.t.*

$\pi_1(Y_{m,l}) = 1$. Moreover, If $l \geq 1$ (resp. m is even and $l \not\equiv 0 \pmod{16}$), then $Y_{m,l}$ is minimal (resp. non-spin) symplectic 4-manifold.

As a consequence, the Lefschetz fibrations in Theorem 1.2, Corollary 1.3 and 1.4 are non-holomorphic (Corollary 3.4).

We have the following natural question: what Lefschetz fibrations satisfy the slope inequality? By combining the results of [35], [36] and [23], we can show that Lefschetz fibrations with $b_2^+ = 1$ satisfy the slope inequality. Stipsicz showed that if $X \rightarrow S^2$ is a relatively minimal genus- g Lefschetz fibration over S^2 with $b_2^+(X) = 1$ and X is not diffeomorphic to the blow-up of a ruled surface (i.e., diffeomorphic to a S^2 -bundle over Σ_k), then $b_1(X) \in \{0, 2\}$ and $e \geq 0$ (see [35], Corollary 3.3 and 3.5). In particular, if X is the blow-up of a S^2 -bundle over Σ_k , then $k \leq g/2$ (see [23], Proposition 4.4). Then, we obtain the following result.

Theorem 1.5. *Let $f : X \rightarrow S^2$ be a genus- g Lefschetz fibration with $b_2^+(X) = 1$ for $g \geq 2$. Suppose that X is not diffeomorphic to the blow-up of a ruled surface.*

(1) *If $b_1(X) = 0$, then $4 - 4/g \leq \lambda_f \leq 8 + 1/g$.*

(2) *If $b_1(X) = 2$, then $4 - 4/g \leq \lambda_f \leq 8$.*

Suppose that X is the blow-up of a S^2 -bundle over Σ_k ($0 \leq k \leq g/2$).

(3) *$(4 - 4/g \leq) 4 + 4(k - 1)/(g - k) \leq \lambda_f \leq 8$. This lower bound is sharp.*

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2. PRELIMINARIES

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let Γ_g be the mapping class group of Σ_g . We denote by t_c the right handed Dehn twist about a simple closed curve c on an oriented surface. $t_c t_d$ means that we first apply t_d then t_c .

We begin by recalling the definition and basic properties of Lefschetz fibrations. (More details can be found in [15].)

Definition 2.1. Let X be a closed, oriented smooth 4-manifold. A smooth map $f : X \rightarrow S^2$ is a genus- g Lefschetz fibration if it satisfies the following condition :

- (i) f has finitely many critical values $b_1, \dots, b_n \in S^2$, and f is a smooth Σ_g -bundle over $S^2 - \{b_1, \dots, b_n\}$,
- (ii) for each i ($i = 1, \dots, n$), there exists a unique critical point p_i in the *singular fiber* $f^{-1}(b_i)$ such that about each b_i and $f^{-1}(b_i)$ there are complex local coordinate charts agreeing with the orientations of X and S^2 on which f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$,
- (iii) f is relatively minimal (i.e. no fiber contains a (-1) -sphere.)

A Lefschetz fibration $f : X \rightarrow S^2$ is *holomorphic* if there are complex structures on both X and S^2 with holomorphic projection f .

Each singular fiber is obtained by collapsing a simple closed curve (the *vanishing cycle*) in the regular fiber. The monodromy of the fibration around a singular fiber is given by a right handed Dehn twist along the corresponding vanishing cycle.

Once we fix an identification of Σ_g with the fiber over a base point of S^2 , we can characterize the Lefschetz fibration $f : X \rightarrow S^2$ by its *monodromy representation* $\pi_1(S^2 - \{b_1, \dots, b_n\}) \rightarrow \Gamma_g$. Let $\gamma_1, \dots, \gamma_n$ be an ordered system of generating loops for $\pi_1(S^2 - \{b_1, \dots, b_n\})$, such that each γ_i encircles only b_i and $\prod \gamma_i$ is homotopically trivial. Thus, the monodromy of f comprises a factorization

$$t_{v_1} t_{v_2} \cdots t_{v_n} = 1 \in \Gamma_g,$$

where v_i are vanishing cycles of the singular fibers. This factorization is called the *positive relator*.

According to theorems of Kas [18] and Matsumoto [25], if $g \geq 2$, then the isomorphism class of a Lefschetz fibration is determined by a positive relator modulo simultaneous conjugations

$$t_{v_1} t_{v_2} \cdots t_{v_n} \sim t_{\phi(v_1)} t_{\phi(v_2)} \cdots t_{\phi(v_n)} \quad \text{for all } \phi \in \Gamma_g$$

and elementary transformations

$$\begin{aligned} t_{v_1} \cdots t_{v_i} t_{v_{i+1}} t_{v_{i+2}} \cdots t_{v_n} &\sim t_{v_1} \cdots t_{v_{i+1}} t_{t_{v_{i+1}}^{-1}(v_i)} t_{v_{i+2}} \cdots t_{v_n}, \\ t_{v_1} \cdots t_{v_i} t_{v_{i+1}} t_{v_{i+2}} \cdots t_{v_n} &\sim t_{v_1} \cdots t_{t_{v_i}(v_{i+1})} t_{v_i} t_{v_{i+2}} \cdots t_{v_n}. \end{aligned}$$

Note that $\phi t_{v_i} \phi^{-1} = t_{\phi(v_i)}$ and $t_{v_{i+1}}^{-1} t_{v_i} t_{v_{i+1}} = t_{t_{v_{i+1}}^{-1}(v_i)}$. For all $\phi \in \Gamma_g$, let $\phi(\varrho)$ be the positive relator which is obtained by applying simultaneous conjugations by ϕ to a positive relator ϱ . We denote a Lefschetz fibration associated to a positive relator $\varrho \in \Gamma_g$ by $f_\varrho : X_\varrho \rightarrow S^2$. Clearly, if $\varrho_1 \sim \varrho_2$ in Γ_g (i.e. ϱ_2 is obtained by applying elementary transformations or simultaneous conjugations to ϱ_1), then $\chi_{f_{\varrho_1}} = \chi_{f_{\varrho_2}}$ and $K_{f_{\varrho_1}}^2 = K_{f_{\varrho_2}}^2$.

For positive relators ϱ_1 and ϱ_2 in Γ_g , the genus- g Lefschetz fibration $f_{\varrho_1 \varrho_2} : X_{\varrho_1 \varrho_2} \rightarrow S^2$ is the (trivial) fiber sum of f_{ϱ_1} and f_{ϱ_2} . Since $\sigma(X_{\varrho_1 \varrho_2}) = \sigma(X_{\varrho_1}) + \sigma(X_{\varrho_2})$ and $e(X_{\varrho_1 \varrho_2}) = e(X_{\varrho_1}) + e(X_{\varrho_2}) + 4(g-1)$, we see $\chi_{f_{\varrho_1 \varrho_2}} = \chi_{f_{\varrho_1}} + \chi_{f_{\varrho_2}}$ and $K_{f_{\varrho_1 \varrho_2}}^2 = K_{f_{\varrho_1}}^2 + K_{f_{\varrho_2}}^2$. In particular, if $\varrho_1 \sim \varrho_2$, then $\chi_{f_{\varrho_1 \varrho_2}} = 2\chi_{f_{\varrho_1}} (= 2\chi_{f_{\varrho_2}})$ and $K_{f_{\varrho_1 \varrho_2}}^2 = 2K_{f_{\varrho_1}}^2 (= 2K_{f_{\varrho_2}}^2)$.

We next begin with a definition of the lantern relation (see [7], [17]).

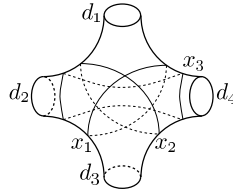


FIGURE 1. The curves $d_1, d_2, d_3, d_4, x_1, x_2, x_3$.

Definition 2.2. Let Σ_0^4 denote a sphere with 4 boundary components. Let d, d_1, d_2, d_3, d_4 be the 4 boundary curves of Σ_0^4 and let x_1, x_2, x_3 be the interior curves as shown in Figure 1. Then, we have the *lantern relation*

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} = t_{x_1} t_{x_2} t_{x_3}.$$

Let ϱ be a positive relator of Γ_g . Let $d_1, d_2, d_3, d_4, x_1, x_2, x_3$ be curves as in Definition 2.2. Suppose that ϱ includes $t_{d_1} t_{d_2} t_{d_3} t_{d_4}$ as a subword :

$$\varrho = U \cdot t_{d_1} t_{d_2} t_{d_3} t_{d_4} \cdot V,$$

where U and V are products of right handed Dehn twists. Then, by the lantern relation, the product of right handed Dehn twists

$$\varrho' = U \cdot t_{x_1} t_{x_2} t_{x_3} \cdot V$$

is also a positive relator of Γ_g .

This operation is one of substitution techniques introduced by Fuller.

Definition 2.3. We say that ϱ' is obtained by applying an *L-substitution* to ϱ . Conversely, ϱ is said to be obtained by applying an *L^{-1} -substitution* to ϱ' . We also call these two kinds of operations *lantern substitutions*.

Proposition 2.4 (Endo and Nagami, [11], Theorem 4.3 and Proposition 3.12). *Let ϱ, ϱ' be positive relators of Γ_g and let $X_\varrho, X_{\varrho'}$ be the corresponding Lefschetz fibrations over S^2 , respectively. Suppose that ϱ is obtained by applying an L^{-} -substitution to ϱ' . Then, $\sigma(X_\varrho) = \sigma(X_{\varrho'}) - 1$ and $e(X_\varrho) = e(X_{\varrho'}) + 1$. Therefore,*

$$\chi_{f_\varrho} = \chi_{f_{\varrho'}}, \quad K_{f_\varrho}^2 = K_{f_{\varrho'}}^2 - 1.$$

Remark 2.5. Endo and Gurtas [9] showed that $X_{\varrho'}$ is a rational blowdown of X_ϱ introduced by Fintushel and Stern [12]. Such relations were also generalized by Endo, Mark, and Van Horn-Morris [10].

3. PROOFS OF MAIN RESULTS

In order to prove Theorem 1.2, we will need the following positive relator:

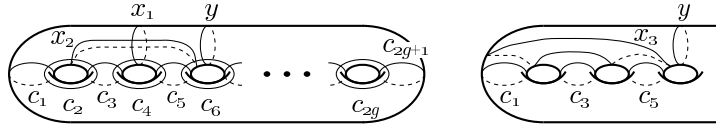


FIGURE 2. The curves $c_1, \dots, c_{2g+1}, x_1, x_2, x_3, y$.

Suppose $g \geq 3$. Let $c_1, c_2, \dots, c_{2g+1}$ be the curves in Σ_g as shown in Figure 2. We denote by $h_g (\in \Gamma_g)$ the product of $8g + 4$ right handed Dehn twists

$$h_g := (t_{c_1} t_{c_2} \cdots t_{c_{2g+1}}^2 \cdots t_{c_2} t_{c_1})^2.$$

It is well known that h_g is a positive relator in Γ_g and that $\sigma(X_{h_g}) = -4(g + 1)$ and $e(X_{h_g}) = 4(g + 2)$. This gives $\chi_{f_{h_g}} = g$, $K_{f_{h_g}}^2 = 4g - 4$ and $\lambda_{f_{h_g}} = 4 - 4/g$ (i.e. f_{h_g} is lying on the slope line). By Lemma 3.2 (c) of [1], we have $\pi_1(X_{h_g}) = 1$.

Proof of Theorem 1.2. Let x_1, x_2, x_3, y be the curves as shown in Figure 2. Since c_1, x_i are nonseparating curves, there exists a diffeomorphism f_i such that $\phi_i(c_1) = x_i$. Hence, we have the following positive relator r_i ($i = 1, 2, 3$):

$$\begin{aligned} r_i &= \phi_i h_g \phi_i^{-1} = \phi_i (t_{c_1} t_{c_2} \cdots t_{c_{2g+1}}^2 \cdots t_{c_2} t_{c_1})^2 \phi_i^{-1} \\ &= (t_{\phi_i(c_1)} t_{\phi_i(c_2)} \cdots t_{\phi_i(c_{2g+1})}^2 \cdots t_{\phi_i(c_2)} t_{\phi_i(c_1)})^2 \\ &= (t_{x_i} t_{\phi_i(c_2)} \cdots t_{\phi_i(c_{2g+1})}^2 \cdots t_{\phi_i(c_2)} t_{\phi_i(c_1)})^2 = 1 \in \Gamma_g. \end{aligned}$$

Let $r'_g = r_1 r_2 r_3$. Since $f_{r'_g}$ is the fiber sum of f_{r_1} , f_{r_2} and f_{r_3} which are obtained by applying simultaneous conjugations to h_g , we have

$$\chi_{f_{r'_g}} = 3\chi_{f_{h_g}} = 3g, \quad K_{f_{r'_g}}^2 = 3K_{f_{h_g}}^2 = 3(4g - 4).$$

We apply elementary transformations to r'_g as follows:

$$\begin{aligned} r'_g &= r_1 r_2 r_3 \\ &= t_{x_1} t_{\phi_1(c_2)} \cdots t_{\phi_1(c_2)} \underline{t_{\phi_1(c_1)} \cdot t_{x_2} t_{\phi_2(c_2)} \cdots t_{\phi_2(c_1)} \cdot t_{x_3} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)}} \\ &\sim t_{x_1} t_{\phi_1(c_2)} \cdots t_{\phi_1(c_2)} \underline{t_{x_2} t_{t_{x_2}^{-1}(\phi_1(c_1))} t_{\phi_2(c_2)} \cdots t_{\phi_2(c_1)} \cdot t_{x_3} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)}} \\ &\vdots \\ &\sim t_{x_1} t_{x_2} t_{t_{x_2}^{-1}(\phi_1(c_2))} \cdots t_{t_{x_2}^{-1}(\phi_1(c_2))} t_{t_{x_2}^{-1}(\phi_1(c_1))} t_{\phi_2(c_2)} \cdots t_{\phi_2(c_1)} \cdot t_{x_3} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)} \\ &\sim t_{x_1} t_{x_2} t_{t_{x_2}^{-1}(\phi_1(c_2))} \cdots t_{t_{x_2}^{-1}(\phi_1(c_2))} t_{t_{x_2}^{-1}(\phi_1(c_1))} t_{\phi_2(c_2)} \cdots t_{\phi_2(c_1)} \cdot \underline{t_{x_3} t_{t_{x_3}^{-1}(\phi_2(c_1))} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)}} \\ &\vdots \\ &\sim (t_{x_1} t_{x_2} t_{x_3}) W, \end{aligned}$$

where W is a product of $24g + 9$ right handed Dehn twists. By the lantern relation, we get the following positive relator r_g :

$$r_g := (t_{c_1} t_{c_3} t_{c_5} t_y) W.$$

Since r_g is obtained by applying an L^{-1} -substitution to r'_g , by Proposition 2.4

$$\chi_{f_{r_g}} = 3g, \quad K_{f_{r_g}}^2 = 3(4g - 4) - 1.$$

Then, the slope of f_{r_g} is equal to $4 - 4/g - 1/3g$.

Since it is easy to check that W includes the Dehn twist about a curve $\phi_3(c_i)$ for $1 \leq i \leq 2g + 1$, $\pi_1(X_{r_g}) = \pi_1(X_{h_g}) = 1$ by Lemma 3.2 (c) of [1]. From Theorem of [31] and $\sigma(X_{r_g}) = 3(-4(g + 1)) - 1$, we see that X_{r_g} is non-spin. This completes the proof of Theorem 1.2. \square

Remark 3.1. Since r_g is obtained by applying an L^{-} -substitution to r'_g , X_{r_g} is a *rational blowup* of $X_{r'_g}$. By applying elementary transformations to a relator corresponding to a Lefschetz fibration which is obtained by taking a twisted fiber sum with sufficiently many Lefschetz fibrations, we obtain a positive relator such that we can apply a monodromy substitution, which corresponds to the operation of rational blowdown (resp. rational blowup) in [10], to it.

Remark 3.2. Miyachi and Shiga [26] produced genus- g Lefschetz fibrations over Σ_{2m} which do not satisfy the slope inequality.

Proof of Corollary 1.3. For any $m \geq 0$, we consider the Lefschetz fibration $f_{r_g h_g^m} : X_{r_g h_g^m} \rightarrow S^2$ which is the fiber sum of f_{r_g} and m copies of f_{h_g} . Then,

$$\begin{aligned}\chi_{f_{r_g h_g^m}} &= \chi_{f_{r_g}} + m\chi_{f_{h_g}} = (3+m)g, \\ K_{f_{r_g h_g^m}}^2 &= K_{f_{r_g}}^2 + mK_{f_{h_g}}^2 = (3+m)(4g-4) - 1.\end{aligned}$$

Therefore, we obtain $\lambda_{f_{r_g h_g^m}} = 4 - 4/g - 1/(m+3)g$.

Let $f_{m,l} : X_{m,l} \rightarrow S^2$ be the fiber sum of l copies of $f_{r_g h_g^m}$ (i.e. $f_{m,l} = f_{(r_g h_g^m)^l}$). By $\lambda_{f_{r_g h_g^m}} = 4 - 4/g - 1/(m+3)g$ and $\pi_1(X_{r_g}) = 1$, we have $\lambda_{f_{m,l}} = 4 - 4/g - 1/(m+3)g$ and $\pi_1(X_{m,l}) = 1$. By the result of Usher [37], $X_{m,l}$ is minimal for $(m,l) \neq (0,0)$. From Rohlin's Theorem and $\sigma(X_{m,l}) = l\{-4(g+1)(m+3) - 1\}$, for $l \not\equiv 0 \pmod{16}$, $X_{m,l}$ is non-spin. This completes the proof. \square

Proof of Corollary 1.4. Let $\varrho_1 = h_g$ and $\varrho_2 = r_g$. When we apply the argument of Theorem 1.2 again, with $\varrho_1 = h_g$ replaced by $\varrho_2 = r_g$, we obtain a genus- g Lefschetz fibration $f_{\varrho_3} : X_{\varrho_3} \rightarrow S^2$ with

$$\begin{aligned}\chi_{f_{\varrho_3}} &= 3\chi_{f_{\varrho_2}} = 3 \cdot 3\chi_{f_{\varrho_1}} \\ K_{f_{\varrho_3}}^2 &= 3K_{f_{\varrho_2}}^2 - 1 = 3(3K_{f_{\varrho_1}} - 1) - 1.\end{aligned}$$

By repeating this argument, we get a genus- g Lefschetz fibration f_{ϱ_m} ($m \geq 1$) with

$$\begin{aligned}\chi_{f_{\varrho_m}} &= 3^{m-1}\chi_{f_{\varrho_1}} = 3^{m-1}g \\ K_{f_{\varrho_m}}^2 &= 3(\cdots(3(3K_{f_{\varrho_1}}^2 - 1) - 1)\cdots) - 1 = 3^{m-1}K_{f_{\varrho_1}} - 3^{m-2} - \cdots - 3 - 1 \\ &= 3^{m-1}(4g-4) - (3^{m-1} - 1)/2.\end{aligned}$$

Therefore, $\lambda_{f_{\varrho_m}} = 4 - 4/g - 1/2g + 1/(2 \cdot 3^{m-1}g)$.

Let $f'_{m,l} : Y_{m,l} \rightarrow S^2$ be the fiber sum of l copies of f_{ϱ_m} , and so $\lambda_{f'_{m,l}} = 4 - 4/g - 1/2g + 1/(2 \cdot 3^{m-1}g)$. By an argument similar to the proof of Theorem 1.2, we see $\pi_1(Y_{m,l}) = 1$. By the result of Usher, $Y_{m,l}$ is minimal for $l \geq 1$. From Rohlin's Theorem and $\sigma(Y_{m,l}) = l\{3^{m-1}(-4(g+1)) - (3^{m-1} - 1)/2\}$, $Y_{m,l}$ is non-spin if m is even and $l \not\equiv 0 \pmod{16}$. This completes the proof. \square

From the slope inequality for holomorphic fibrations, we have the following necessary condition for a Lefschetz fibration to be holomorphic :

Proposition 3.3 (Xiao, [38]). *If a Lefschetz fibration f is holomorphic, then the slope inequality $\lambda_f \geq 4 - 4/g$ holds.*

As a consequence, we have the following results.

Corollary 3.4. *The Lefschetz fibrations of Theorem 1.2, Corollary 1.3 and 1.4 are non-holomorphic.*

Remark 3.5. There are various kinds of non-holomorphic Lefschetz fibrations. By fiber summing two copies of genus-2 Lefschetz fibration due to Matsumoto [25], Ozbagci and Stipsicz [27] constructed non-holomorphic genus-2 Lefschetz fibrations whose total space does not admit a complex structure. Korkmaz [22] generalized their examples to $g \geq 3$. The above examples of Fintushel and Stern are also non-holomorphic Lefschetz fibrations. From study of divisors in moduli space, Smith [34] showed that a genus-3 Lefschetz fibration over S^2 which was produced by

Fuller is non-holomorphic. Endo and Nagami [11] constructed some examples of non-holomorphic Lefschetz fibrations which violate lower bounds of the slope for non-hyperelliptic fibrations of genus 3, 4 and 5 from the results of Konno [19], [20] and Chen [6]. Hirose [16] also gave some examples of $g = 3, 4$.

Proof of Theorem 1.5. Let $f : X \rightarrow S^2$ be a nontrivial genus- g Lefschetz fibration with $b_2^+(X) = 1$. Note that $-4(g-1) \leq K^2$, and so $4(g-1) \leq K_f^2$ (see [35], Lemma 3.2). Suppose that X is not diffeomorphic to the blow-up of a ruled surface.

First, suppose $b_1 = 0$. Since $b_2^+ = 1$ and $\chi_f = (\sigma + e)/4 + (g-1) = (b_2^+ - b_1 + 1)/2 + (g-1) = g$, we have $4(g-1)/g \leq K_f^2/\chi_f = \lambda_f$. On the other hand, since $K^2 = 3\sigma + 2e = 5b_2^+ - b_2^- + 4 - 4b_1 = 9 - b_2^-$, by $b_2^- \geq 0$, we have $\lambda_f = K_f^2/\chi_f = \{9 - b_2 + 8(g-1)\}/g \leq 8 + 1/g$.

Next, suppose $b_1 = 2$. Then, $\chi_f = g-1$. Therefore, by $4(g-1) \leq K_f^2$, we have $4 \leq \lambda_f$. Since $0 \leq e = 2 - 2b_1 + b_2^+ + b_2^- = 2 - 4 + 1 + b_2^- = -1 + b_2^-$, we obtain $\lambda_f = \{1 - b_2^- + 8(g-1)\}/(g-1) \leq 8$.

Finally, suppose that X is the m -fold blow-up of a S^2 -bundle over Σ_k . Let Y be the S^2 -bundle over Σ_k . Then, since $b_1(Y) = 2k$, $b_2^\pm(Y) = 1$ and $X = Y \# m\overline{\mathbb{CP}^2}$, we have $b_1(X) = 2k$, $b_2^+(X) = 1$, $b_2^-(X) = m+1$, $e(X) = 4 - 4k + m$ and $\sigma(X) = -m$. Hence, we have $\lambda_f = 8 - m/(g-k)$. From $m \geq 0$, $\lambda_f \leq 8$. We will give lower bounds for λ_f . By Lemma 3.2 in [36], $4(2k-g) + m \leq 4$. This gives $2k \leq g$ or $2k = g+1$ and $m = 0$. By Proposition 4.4 in [23], we need only consider $2k \leq g$. From $\lambda_f = 8 - m/(g-k)$, $4(2k-g) + m \leq 4$ and $0 \leq k \leq g/2$, we have $\lambda_f \geq 4 + 4(k-1)/(g-k)$. Fintushel and Stern [14] showed that $(\Sigma_k \times S^2) \# 4m\overline{\mathbb{CP}^2}$ admits a genus- $(2k+m-1)$ Lefschetz fibration f_{FS} over S^2 . When $m = g-2k+1$, we find $b_2^+ = 1$ and that $\lambda_{f_{FS}} = 4 + 4(k-1)/(g-k)$. This completes the proof. \square

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